

UNIVERSAL REALISATORS FOR HOMOLOGY CLASSES

ALEXANDER A. GAIFULLIN

ABSTRACT. We study oriented closed manifolds M^n possessing the following *Universal Realisation of Cycles (URC) Property*: For each topological space X and each homology class $z \in H_n(X, \mathbb{Z})$, there exist a finite-sheeted covering $\widehat{M}^n \rightarrow M^n$ and a continuous mapping $f : \widehat{M}^n \rightarrow X$ such that $f_*[\widehat{M}^n] = kz$ for a non-zero integer k . We find wide class of examples of such manifolds M^n among so-called small covers of simple polytopes. In particular, we find 4-dimensional hyperbolic manifolds possessing the URC property. As a consequence, we obtain that for each 4-dimensional oriented closed manifold N^4 , there exists a mapping of non-zero degree of a hyperbolic manifold M^4 to N^4 . This was conjectured by Kotschick and Löh.

1. INTRODUCTION

Unless otherwise stated, all manifolds are supposed to be connected, compact, without boundary, and oriented.

Let M^n and N^n be manifolds of the same dimension n . We say that M^n *dominates* N^n and write $M^n \geq N^n$ if there exists a non-zero degree mapping $M^n \rightarrow N^n$. Domination is a transitive relation on the set of homotopy types of connected oriented closed manifolds. This relation goes back to the works of Milnor and Thurston [19] and Gromov [15] and was first explicitly defined in the paper of Carlson and Toledo [5] with a reference to a lecture of Gromov. Good surveys of results on domination relation can be found in [27] for the 3-dimensional case and in [17] for an arbitrary dimension. Obviously, $M^n \geq S^n$ for every n -dimensional manifold M^n . Hence the sphere S^n is the minimal element with respect to the domination relation. We shall focus on the following question of Carlson and Toledo [5] and the conjectural answer to it of Kotschick and Löh [17].

Question 1.1 (Carlson–Toledo, 1989). *Is there an easily describable maximal class of homotopy types with respect to the domination relation, that is, a collection \mathcal{C}_n of n -dimensional manifolds such that given any manifold N^n , there exists an $M^n \in \mathcal{C}_n$ satisfying $M^n \geq N^n$?*

Recall that a *hyperbolic manifold* is a manifold admitting a Riemannian metric of constant negative sectional curvature.

Conjecture 1.2 (Kotschick–Löh, 2008). *In every dimension $n \geq 2$, closed oriented hyperbolic manifolds represent a maximal class of homotopy types with respect to the domination relation.*

This conjecture is trivially true for $n = 2$. By a result of Brooks [2], it is also true for $n = 3$. Besides, Kotschick and Löh [17] showed that it is true for 4-dimensional manifolds with finite fundamental groups. One of the goals of the present paper is to prove Conjecture 1.2 for $n = 4$.

The work was partially supported by RFBR (project 11-01-00694), by a grant of the President of Russian Federation (project MD-4458.2012.1), by a grant of the Government of the Russian Federation (project 2010-220-01-077), and by a programme of the Branch of Mathematical Sciences of the Russian Academy of Sciences.

The following answer to Question 1.1 was obtained by the author [12], [13] in 2008. In every dimension n , there is a manifold M_0^n such that each manifold N^n is dominated by a finite-sheeted covering \widehat{M}_0^n of M_0^n . Thus we can take for \mathcal{C}_n the collection of all finite-sheeted coverings of M_0^n . The manifold M_0^n is a so-called *small cover* of the permutahedron (see sections 2 and 3 for definitions). By a result of Tomei [24], M_0^n is homeomorphic to an isospectral manifold of $(n+1) \times (n+1)$ tridiagonal symmetric real matrices, which is important for integrable systems, since it carries the famous Toda flow. However, for $n \geq 3$, the manifold M_0^n is not hyperbolic, since $\pi_1(M_0^n)$ contains a free Abelian subgroup of rank 2. We say that M^n *virtually dominates* N^n if there exists a finite-sheeted covering \widehat{M}^n of M^n such that $\widehat{M}^n \geq N^n$. Then the author's result in [12], [13] can be formulated as follows.

Theorem 1.3. *The manifold M_0^n is maximal with respect to the virtual domination relation, i. e., it virtually dominates every n -dimensional manifold N^n .*

Gromov [16] suggested a *hyperbolisation procedure* that for each polyhedron P , yields a polyhedron P_h and a mapping $f : P_h \rightarrow P$ such that P_h has a metric of non-positive curvature in sense of Alexandrov (see definition in section 3), f induces an injection in integral cohomology, and singularities of P_h are “not worse” than singularities of P . In particular, if P is a PL manifold, then P_h is also a PL manifold and $f : P_h \rightarrow P$ is a degree 1 mapping. Besides, f takes the rational Pontryagin classes of P to the rational Pontryagin classes of P_h . This result of Gromov implies that each manifold can be dominated (with degree 1) by a manifold of non-positive curvature. Gromov's hyperbolisation procedure has been improved by Davis–Januszkiewicz [11] and Charney–Davis [6]. The strongest result has recently been obtained by Ontaneda [21], who has proved that, for a smooth manifold P , P_h can be constructed to be a Riemannian manifold of negative sectional curvature in an arbitrarily small interval $[-1-\varepsilon, -1]$. However, even the class of all Riemannian manifolds with such pinched sectional curvature seems to be too wide for Question 1.1. Theorem 1.3 shows that more narrow classes \mathcal{C}_n can be found.

Until now most results on domination relation in dimensions $n > 3$ have been negative. For example, it is known that there exist manifolds that cannot be dominated by a Kähler manifold [5] and there exist manifolds that cannot be dominated by a direct product $M_1 \times M_2$ of two manifolds of positive dimensions [17].

The domination relation is closely related to Steenrod's problem on realisation of cycles. The classical question of Steenrod is as follows. Given a homology class $z \in H_n(X, \mathbb{Z})$ of a topological space X , does there exist a smooth manifold M^n and a continuous mapping $f : M^n \rightarrow X$ such that $f_*[M^n] = z$? If such M^n and f exist, we shall say that z is *realisable*. A well-known theorem of Thom [23] claims that there exist non-realisable homology classes, but, for each z , a certain multiple kz , $k \in \mathbb{Z}_{>0}$, is realisable. An important question is to describe a class \mathcal{C}_n of n -dimensional manifolds such that each n -dimensional homology class of each space X can be realised as an image of the fundamental class of a manifold belonging to \mathcal{C}_n . Thom's theorem easily implies that this question is equivalent to Question 1.1. In particular, a manifold M^n is maximal with respect to the virtual domination relation if and only if it has the following *Universal Realisation of Cycles (URC) Property*.

Property URC. *For each X and each $z \in H_n(X, \mathbb{Z})$, there exist a finite-sheeted covering \widehat{M}^n of M^n and a mapping $f : \widehat{M}^n \rightarrow X$ such that $f_*[\widehat{M}^n] = kz$ for a non-zero integer k .*

Manifolds satisfying this condition will be called *URC-manifolds*. By Theorem 1.3, URC-manifolds exist in every dimension. It is an interesting problem to describe the class of URC-manifolds. The following proposition is straightforward.

Proposition 1.4. 1) Suppose \widehat{M}^n is a finite-sheeted covering of M^n ; then \widehat{M}^n is a URC-manifold if and only if M^n is a URC-manifold.

2) Suppose M^n is a URC-manifold; then the connected sum $M^n \# N^n$ is a URC-manifold for every N^n .

3) Suppose N^n is a URC-manifold; then M^n is a URC-manifold if and only if M^n virtually dominates N^n .

In this paper, our goal is to obtain a wide collection of examples of URC-manifolds. These examples are small covers of simple polytopes satisfying certain special conditions. In particular, in dimensions $n \leq 4$ we shall find hyperbolic URC-manifolds. Thus we shall prove Conjecture 1.2 for $n = 4$.

We denote by \mathbb{S}^n , \mathbb{R}^n , and \mathbb{H}^n the spaces of constant curvature $+1$, 0 , and -1 respectively, i. e., the round sphere of radius 1, the Euclidean space, and the Lobachevsky space.

Each compact hyperbolic manifold M^n is isometric to the quotient of \mathbb{H}^n by the action of a torsion-free uniform discrete transformation group $\Gamma \subset \text{Isom}(\mathbb{H}^n)$. Recall that two discrete subgroups $\Gamma_1, \Gamma_2 \subset \text{Isom}(\mathbb{H}^n)$ are said to be *commensurable* if $\Gamma_1 \cap \Gamma_2$ has finite indices in both Γ_1 and Γ_2 . A discrete subgroup $W \subset \text{Isom}(\mathbb{H}^n)$ is called a *reflection subgroup* if it is generated by orthogonal reflections in hyperplanes $H \subset \mathbb{H}^n$. A reflection subgroup is called *right-angular* if the mirrors of any two reflections $s, s' \in W$ either are orthogonal to each other or do not intersect.

Theorem 1.5. Let $M^n = \mathbb{H}^n/\Gamma$ be a compact hyperbolic manifold such that the group $\pi_1(M^n) = \Gamma$ is commensurable with a uniform right-angular reflection subgroup $W \subset \text{Isom}(\mathbb{H}^n)$. Then M^n is a URC-manifold.

Uniform right-angular reflection subgroups $W \subset \text{Isom}(\mathbb{H}^n)$ correspond to compact right-angular polytopes $P \subset \mathbb{H}^n$. (It is well known that every compact right-angular polytope is simple.) Compact right-angular polytopes $P \subset \mathbb{H}^n$ exist for $n = 2, 3, 4$. For example, there exist the right-angular regular dodecahedron in \mathbb{H}^3 and the right-angular regular 120-cell in \mathbb{H}^4 . Many examples of compact right-angular polyhedra in \mathbb{H}^3 were obtained by Löbell [18]. Hence, for $n = 2, 3, 4$, there exist uniform right-angular reflection subgroups $W \subset \text{Isom}(\mathbb{H}^n)$. By Selberg's lemma each such subgroup W contains a finite index torsion-free subgroup Γ . Then \mathbb{H}^n/Γ is a hyperbolic URC manifold.

Corollary 1.6. In dimensions $n = 2, 3, 4$, there exist hyperbolic URC-manifolds. Consequently, each manifold N^n can be dominated by a hyperbolic manifold.

Notice that even in dimension 3 this assertion is stronger than a result of Brooks [2] because the hyperbolic manifolds obtained by the Brooks construction are not coverings of the same manifold.

By a result of Vinberg [26], for $n \geq 5$, there are no compact right-angular polytopes $P \subset \mathbb{H}^n$ and, hence, there are no uniform right-angular reflection subgroups $W \subset \text{Isom}(\mathbb{H}^n)$. Thus Theorem 1.5 cannot be used for $n \geq 5$.

This paper is organised in the following way. In section 2 we describe several constructions of manifolds glued of simple polytopes, in particular, the construction of so-called *small covers* of simple polytopes due to Davis and Januszkiewicz [10]. These constructions play key role throughout the paper. Main results are formulated in section 3. They

include several sufficient conditions for small covers of simple polytopes to be URC-manifolds. In particular, we prove that a small cover of a polytope is a URC-manifold whenever it admits an equivariant metric of strictly negative curvature in sense of Alexandrov (with at least one “smooth” point). The proofs of these results as well as the proof of Theorem 1.5 are given in sections 6 and 7.

Sections 4 and 5 are devoted to the proof of Theorem 1.3. Actually, this theorem has been proved by the author in [12], [13]. However, in the present paper we give a new interpretation of this proof based on a certain group-theoretic construction that essentially clarifies the proof.

The author is grateful to V. M. Buchstaber and E. B. Vinberg for useful comments.

2. MANIFOLDS GLUED OF SIMPLE POLYTOPES

We denote by $[m]$ the set $\{1, 2, \dots, m\}$.

Let S be a finite set and let G be a graph on the vertex set S . Recall that a *right-angular Coxeter group* associated to G is the group W with the set of generators S such that all relations in W are consequences of relations $s^2 = 1$, $s \in S$, and $st = ts$ whenever $\{s, t\}$ is an edge of G . The pair (W, S) is called a *right-angular Coxeter system*.

The following general construction is due to Vinberg [25]. Let X be a topological space and let X_s be closed subsets of it indexed by elements $s \in S$. For a point $p \in X$, denote by $W(p)$ the subgroup of W generated by all elements $s \in S$ such that $p \in X_s$. Endow W with the discrete topology. Consider an equivalence relation \sim on $X \times W$ such that $(p, g) \sim (p', g')$ if and only if $p = p'$ and $g'g^{-1} \in W(p)$, and denote the quotient space $(X \times W)/\sim$ by $\mathcal{U}(W, X, \{X_s\})$. Denote by $[p, g]$ the point of $\mathcal{U}(W, X, \{X_s\})$ corresponding to the equivalence class of (p, g) . The natural *right* action of W on $X \times W$ is compatible with \sim . Hence the right action of W on $\mathcal{U}(W, X, \{X_s\})$ given by $[p, g] \cdot g' = [p, gg']$ is well defined.

Remark 2.1. Usually in this construction the condition $g'g^{-1} \in W(p)$ is replaced with the condition $g^{-1}g' \in W(p)$ to obtain a *left* action of W on $\mathcal{U}(W, X, \{X_s\})$. However, it is convenient for us to have a *right* action here.

Davis [7], [8], and Davis and Januszkiewicz [10] applied this construction to obtain a wide class of manifolds glued of simple convex polytopes. Here we describe several their constructions.

Let P be a simple convex polytope in either \mathbb{R}^n or \mathbb{H}^n with facets F_1, \dots, F_m . We always assume that P is n -dimensional, i.e., is not contained in a hyperplane, and compact. Recall that a polytope is said to be *simple* if every its vertex is contained in exactly n facets.

1. Let $S = \{s_1, \dots, s_m\}$ and let $\{s_i, s_j\}$ be an edge of G if and only if $F_i \cap F_j \neq \emptyset$. Denote by W_P the right-angular Coxeter group associated to the graph G . Let $X = P$ and $X_{s_i} = F_i$, $i = 1, \dots, m$. Put $\mathcal{U}_P = \mathcal{U}(W_P, P, \{F_i\})$. Then

$$\mathcal{U}_P = (P \times W_P)/\sim,$$

where $(p, g) \sim (p', g')$ if and only if $p = p'$ and $g'g^{-1}$ belongs to the subgroup $W(p) \subset W_P$ generated by all s_i such that $p \in F_i$.

2. Let S , X , X_{s_i} be as in the previous construction and let G be a complete graph on the vertex set S . Then the corresponding right-angular Coxeter group is \mathbb{Z}_2^m . We shall use the multiplicative notation for \mathbb{Z}_2^m and we shall denote its generators by a_1, \dots, a_m to avoid confusion with the generators s_1, \dots, s_m of W_P . Put $\mathcal{R}_P = \mathcal{U}(\mathbb{Z}_2^m, P, \{F_i\})$. Then

$$\mathcal{R}_P = (P \times \mathbb{Z}_2^m)/\sim,$$

where $(p, g) \sim (p', g')$ if and only if $p = p'$ and $g'g^{-1}$ belongs to the subgroup of \mathbb{Z}_2^m generated by all a_i such that $x \in F_i$.

3. Suppose that facets of P are coloured in colours from a finite set. We say that a colouring is *regular* if any two intersecting facets are of distinct colours. Assume that P admits a regular colouring of facets in n colours, which we denote by $1, \dots, n$. Let $S = \{b_1, \dots, b_n\}$, $X = P$, and let X_{b_i} be the union of all facets of colour i . Let G be a complete graph on the vertex set S . Then the corresponding right-angular Coxeter group is \mathbb{Z}_2^n . Put $M_P = \mathcal{U}(\mathbb{Z}_2^n, P, \{X_{b_i}\})$. Then

$$M_P = (P \times \mathbb{Z}_2^n) / \sim,$$

where $(p, g) \sim (p', g')$ if and only if $p = p'$ and $g'g^{-1}$ belongs to the subgroup of \mathbb{Z}_2^n generated by all b_i such that p is contained in a facet of colour i . The manifold M_P is called the *small cover of P induced from a linear model*. Notice that if a regular colouring of facets of P exists, it is unique up to a permutation of colours.

4. This is a generalization of the previous construction. A *characteristic function* for P is a mapping $\lambda : [m] \rightarrow \mathbb{Z}_2^n$ that satisfies the following condition:

The elements $\lambda(i_1), \dots, \lambda(i_n)$ generate \mathbb{Z}_2^n whenever i_1, \dots, i_n are pairwise distinct and $F_{i_1} \cap \dots \cap F_{i_n} \neq \emptyset$.

Given a characteristic function, we can define a *small cover* of P by

$$M_{P,\lambda} = (P \times \mathbb{Z}_2^n) / \sim_\lambda,$$

where $(p, g) \sim_\lambda (p', g')$ if and only if $p = p'$ and $g'g^{-1}$ belongs to the subgroup of \mathbb{Z}_2^n generated by all $\lambda(a_i)$ such that $p \in F_i$. If every $\lambda(i)$ is contained in the basis b_1, \dots, b_n , we obtain the small covering induced from a linear model described above. Notice that there exist polytopes that admit no characteristic functions λ .

Remark 2.2. The manifolds \mathcal{R}_P and $M_{P,\lambda}$ have “complex” analogues that are obtained by replacing the groups \mathbb{Z}_2^m and \mathbb{Z}_2^n with the compact tori T^m and T^n and are called *moment-angle manifolds* and *quasi-toric manifolds* respectively. These manifolds have also been defined by Davis and Januszkiewicz [10]. The theory of moment-angle manifolds and quasi-toric manifolds has been fruitfully developed and has found many applications, see [4] and references therein.

The spaces \mathcal{U}_P , \mathcal{R}_P , and $M_{P,\lambda}$ are glued of copies of the polytope P indexed by elements of the groups W_P , \mathbb{Z}_2^m , and \mathbb{Z}_2^n respectively. Hence these spaces have natural cell decompositions with each n -dimensional cell isomorphic to P . It is not hard to see that each vertex v of each of these decompositions is contained in exactly 2^n n -dimensional cells. Moreover, the corners of these 2^n cells meet at vertex v in a standard way, i. e., as the corners of orthants at the origin of \mathbb{R}^n . Hence \mathcal{U}_P , \mathcal{R}_P , and $M_{P,\lambda}$ are PL manifolds. Besides, the dual cell decompositions \mathcal{U}_P^* , \mathcal{R}_P^* , and $M_{P,\lambda}^*$ are decompositions into cubes.

The manifolds \mathcal{U}_P , \mathcal{R}_P , and $M_{P,\lambda}$ carry natural right actions of the groups W_P , \mathbb{Z}_2^m , and \mathbb{Z}_2^n respectively and

$$\mathcal{U}_P / W_P = \mathcal{R}_P / \mathbb{Z}_2^m = M_{P,\lambda} / \mathbb{Z}_2^n = P.$$

Further, we have surjective homomorphisms $\eta : W_P \rightarrow \mathbb{Z}_2^m$ and $\lambda_* : \mathbb{Z}_2^m \rightarrow \mathbb{Z}_2^n$ given by $\eta(s_i) = a_i$ and $\lambda_*(a_i) = \lambda(i)$ respectively. It can be easily seen that the groups $\ker \eta$ and $\ker \lambda_*$ act freely on \mathcal{U}_P and \mathcal{R}_P respectively and

$$\mathcal{U}_P / \ker \eta = \mathcal{R}_P, \quad \mathcal{R}_P / \ker \lambda_* = M_{P,\lambda}.$$

Thus \mathcal{U}_P is a regular covering of \mathcal{R}_P and \mathcal{R}_P is a 2^{m-n} -sheeted regular covering of $M_{P,\lambda}$. Vice versa, the quotient of \mathcal{R}_P by any subgroup $A \subset \mathbb{Z}_2^m$, $A \cong \mathbb{Z}_2^{m-n}$, that acts freely on it is a small cover of P .

Davis [7] proved that the manifold \mathcal{U}_P is simply connected. Hence \mathcal{U}_P is the universal covering of the manifolds \mathcal{R}_P and $M_{P,\lambda}$. A simple polytope P is called a *flag polytope* if a collection F_{i_1}, \dots, F_{i_r} of its facets has non-empty intersection whenever all pairwise intersections $F_{i_k} \cap F_{i_l}$ are non-empty. Davis [7] proved that if P is a flag polytope, then the manifold \mathcal{U}_P is contractible. Hence, the manifolds \mathcal{R}_P and $M_{P,\lambda}$ are aspherical and their fundamental groups are isomorphic to the groups $\ker \eta$ and $\ker(\lambda_* \eta)$ respectively.

To a simple polytope P with facets F_1, \dots, F_m is assigned a simplicial complex K_P on the vertex set $\{v_1, \dots, v_m\}$ such that vertices v_{i_1}, \dots, v_{i_k} span a simplex in K_P if and only if $F_{i_1} \cap \dots \cap F_{i_k} \neq \emptyset$. The complex K_P is an $(n-1)$ -dimensional combinatorial sphere. (A simplicial complex K is called an $(n-1)$ -dimensional *combinatorial sphere* if it is PL homeomorphic to the boundary of the n -dimensional simplex.) If P is a simple polytope in \mathbb{R}^n , then K_P is isomorphic to the boundary of the dual simplicial polytope P^* .

The constructions of manifolds \mathcal{U}_P , \mathcal{R}_P , and $M_{P,\lambda}$ can be extended to a wider class of objects than simple polytopes. It is well known that not every combinatorial sphere can be realised as the boundary of a simplicial polytope. Nevertheless, it is convenient to assign to every combinatorial sphere K certain object P_K which turns to be the dual simple polytope whenever K is the boundary of a simplicial polytope. The construction is as follows. Suppose K is an $(n-1)$ -dimensional combinatorial sphere on the vertex set $\{v_1, \dots, v_m\}$. Let $P_K = \text{cone}(K')$ be the cone over the barycentric subdivision of K and let F_i be the star of a vertex v_i in K' , that is, a subcomplex of K' consisting of all simplices containing v_i . We say that P_K is the *simple cell* dual to K , F_i are *facets* of P_K , and non-empty intersections of facets are *faces* of P_K . The simple cell P_K has the following properties that mimic properties of simple polytopes:

- P_K is PL homeomorphic to the n -dimensional simplex;
- $\partial P_K = F_1 \cup \dots \cup F_m$;
- $F_{i_1} \cap \dots \cap F_{i_k}$ is PL homeomorphic to the $(n-k)$ -dimensional simplex if the vertices v_{i_1}, \dots, v_{i_k} span a $(k-1)$ -dimensional simplex of K , and $F_{i_1} \cap \dots \cap F_{i_k}$ is empty if the vertices v_{i_1}, \dots, v_{i_k} do not span a simplex of K .

The described constructions $P \mapsto K_P$ and $K \mapsto P_K$ are mutually inverse and yield a one-to-one correspondence between n -dimensional simple cells and $(n-1)$ -dimensional combinatorial spheres. The constructions of the manifolds \mathcal{U}_P , \mathcal{R}_P , and $M_{P,\lambda}$ are immediately extended to the case of an arbitrary simple cell P .

Remark 2.3. Indeed, the requirement that K is a combinatorial sphere is not needed to construct P_K . This construction can be applied to an arbitrary simplicial complex K , and P_K is called a *simple polyhedral complex* dual to K (see [8], [10]). The spaces \mathcal{U}_{P_K} , \mathcal{R}_{P_K} , and $M_{P_K,\lambda}$ can be constructed for an arbitrary simple polyhedral complex P . However, generally they are not manifolds. Still, they are homology manifolds if K is a homology sphere, and they are pseudo-manifolds if K is a pseudo-manifold.

Let us fix terminology and notation concerning simplicial complexes and simple cells. First, we always work with *geometric* simplicial complexes. Second, if K is a simplicial complex, we use the same notation K (instead of $|K|$) for the underlying topological space of K . This causes no ambiguity unless we write $x \in K$. We make a convention that notation $x \in K$ always means that x is a point of the underlying space of K and we never use notation $x \in K$ to indicate that x is a simplex of K . The simplex with vertices u_1, \dots, u_k will be denoted by $[u_1 \dots u_k]$. If K is a combinatorial sphere and P is the dual

simple cell, we always identify with each other the topological spaces ∂P , K' , and K . Now, let K_1 and K_2 be combinatorial spheres and let P_1 and P_2 be the dual simple cells respectively. A simplicial mapping $\varphi : K_1 \rightarrow K_2$ induces the simplicial mapping $\varphi' : K'_1 \rightarrow K'_2$ and the mapping $\bar{\varphi} = \text{cone}(\varphi') : P_1 \rightarrow P_2$. After identifying ∂P_i with K_i , $i = 1, 2$, the mapping $\bar{\varphi}|_{\partial P_1}$ will coincide with the original mapping $\varphi : K_1 \rightarrow K_2$. Let u and v be vertices of K_1 and K_2 respectively and let F and G be the corresponding facets of P_1 and P_2 respectively. It is easy to see that $\bar{\varphi}(F) \subset G$ whenever $\varphi(u) = v$.

Let P_1 and P_2 be simple cells of the same dimension and let $f : P_1 \rightarrow P_2$ be a mapping such that $f(\partial P_1) \subset \partial P_2$. The *degree* of f is, by definition, the degree of the induced mapping $P_1/\partial P_1 \rightarrow P_2/\partial P_2$.

3. URC SMALL COVERS

By definition, the *permutahedron* $\Pi^n \subset \mathbb{R}^{n+1}$ is the convex hull of $(n+1)!$ points $(j_1, j_2, \dots, j_{n+1})$ such that j_1, j_2, \dots, j_{n+1} is a permutation of the set $1, 2, \dots, n+1$. It can be easily seen that Π^n is a simple polytope in the hyperplane $\sum_{i=1}^{n+1} t_i = \frac{(n+1)(n+2)}{2}$, where t_1, \dots, t_n are the standard coordinates in \mathbb{R}^{n+1} . The permutahedron Π^n has $2^{n+1} - 2$ facets F_ω , which are conveniently indexed by non-empty proper subsets $\omega \subset [n+1]$. (A subset $\omega \subset [n+1]$ is called *proper* if $\omega \neq [n+1]$.) The facet F_ω is given by the equation

$$\sum_{i \in \omega} t_i = \sum_{j=n-|\omega|+2}^{n+1} j = \frac{(n+1)(n+2)}{2} - \frac{(n-|\omega|+1)(n-|\omega|+2)}{2}$$

where $|\omega|$ is the cardinality of ω . It is easy to see that facets F_{ω_1} and F_{ω_2} have non-empty intersection if and only if either $\omega_1 \subset \omega_2$ or $\omega_2 \subset \omega_1$. This implies that the simplicial complex K_{Π^n} is isomorphic to the barycentric subdivision of the boundary of the n -simplex.

Colouring each facet F_ω in colour $|\omega|$, we obtain a regular colouring of facets of Π^n in n colours $1, \dots, n$. Hence the small cover M_{Π^n} induced from a linear model is well defined. The following theorem is a reformulation of Theorem 1.3.

Theorem 3.1. *The manifold M_{Π^n} is a URC-manifold.*

This theorem was proved by the author [12], [13] (see also [14] for further applications of this construction). In sections 4 and 5 we shall give a proof of Theorem 3.1. Actually, this is the same proof as the proof given in [12] and [13]. Nevertheless, in the present paper we give a new exposition of this proof using a new group-theoretic lemma, which essentially clarifies the idea behind the construction.

For each characteristic function λ , \mathcal{R}_P is a finite-sheeted covering of $M_{P,\lambda}$. Hence $M_{P,\lambda}$ is a URC-manifold if and only if \mathcal{R}_P is a URC-manifold. In particular, we see that \mathcal{R}_{Π^n} is a URC-manifold and $M_{\Pi^n,\lambda}$ is a URC-manifold for every characteristic function λ . We shall give several sufficient conditions for a manifold \mathcal{R}_P to be a URC-manifold. To formulate the result we need several definitions.

A simplicial complex K is said to be a *flag complex* if a set of vertices of K spans a simplex whenever these vertices are pairwise joined by edges. Another is that K contains “no empty simplices” or satisfies “no- \triangle -condition” (see [16]). Further, we shall say that K does not contain an *empty 4-circuit* if for any pairwise distinct vertices u_1, u_2, u_3, u_4 such that $[u_1u_2]$, $[u_2u_3]$, $[u_3u_4]$, and $[u_4u_1]$ are edges of K , at least one of the diagonals $[u_1u_3]$ and $[u_2u_4]$ is an edge of K . In Gromov’s terminology [16] this condition is called “Siebenmann’s no- \square -condition”.

Let K be an $(n-1)$ -dimensional combinatorial sphere and let $0 < \varepsilon < \pi$. We say that K is ε -fine if there exists a mapping $K \rightarrow \mathbb{S}^{n-1}$ of non-zero degree such that the diameter of the image of every simplex of K is less than ε . A simple cell P is said to be ε -fine if K_P is ε -fine.

Theorem 3.2. *Let P be an n -dimensional simple cell. Assume that P satisfies one of the following conditions:*

- (a) *There exists a simplicial mapping $f : K_P \rightarrow (\partial\Delta^n)'$ of non-zero degree, where $(\partial\Delta^n)'$ is the barycentric subdivision of the boundary of the n -dimensional simplex.*
- (b) *K_P is isomorphic to the barycentric subdivision of a combinatorial sphere L .*
- (c) *P is ε_n -fine, where*

$$\varepsilon_n = \arccos \left(1 - \frac{12}{n(n+1)(n+2)} \right).$$

- (d) *$P \subset \mathbb{H}^n$ is a simple convex polytope whose interior contains a closed ball of radius*

$$\rho_n = \log \left(\sqrt{\frac{n(n+1)(n+2)}{6}} + \sqrt{\frac{n(n+1)(n+2)}{6} - 1} \right).$$

- (e) *K_P is a flag combinatorial sphere without an empty 4-circuit, i. e., K_P satisfies the “no- \triangle -condition” and the “no- \square -condition”.*

Then \mathcal{R}_P is a URC-manifold. Therefore, $M_{P,\lambda}$ is a URC-manifold for every characteristic function λ .

The assertion that each manifold \mathcal{R}_P satisfying condition (x) is a URC-manifold will be referred as Theorem 3.2(x), where x is one of the letters a, b, c, d, and e. Conditions (a), (b), and (e) are purely combinatorial. Notice, however, that condition (a) is harder to check than conditions (b) and (e). Conditions (c) and (d) have more geometric nature. We shall see that (d) \Rightarrow (c) \Rightarrow (a) and (b) \Rightarrow (a). Hence all assertions of Theorem 3.2 except for (e) will follow from assertion (a).

The proof of Theorem 3.2(e) uses theory of $CAT(\kappa)$ spaces. (Good references for this subject are [16] and [1]; for applications to manifolds \mathcal{U}_P , see [9].) Recall the definition of a $CAT(-1)$ space. Let X be a complete geodesic metric space. For every geodesic triangle xyz in X , a *comparison triangle* is a triangle $x^*y^*z^*$ in \mathbb{H}^2 with the same edge lengths. Let p and q be arbitrary points in the boundary of xyz and let p^* and q^* be the corresponding points in the boundary of $x^*y^*z^*$. The *$CAT(-1)$ inequality* is the inequality $\text{dist}_X(p, q) \leq \text{dist}_{\mathbb{H}^2}(p^*, q^*)$. A space X is said to be a *$CAT(-1)$ space* if the $CAT(-1)$ inequality is satisfied for every geodesic triangle xyz and every points p and q in its boundary. A space X is said to have *curvature $K \leq -1$* if the $CAT(-1)$ inequality is satisfied locally, i. e., in a neighbourhood of every point. Replacing in these definitions the Lobachevsky plane \mathbb{H}^2 with the Euclidean plane \mathbb{R}^2 one will obtain a definitions of a $CAT(0)$ space and of a space of curvature $K \leq 0$.

Theorem 3.3. *Assume that the manifold \mathcal{R}_P admits a \mathbb{Z}_2^m -invariant metric of curvature $K \leq -1$ such that there exists a point $o \in \mathcal{R}_P$ whose neighbourhood is isometric to an open subset of \mathbb{H}^n . Then \mathcal{R}_P is a URC-manifold.*

The universal covering of a space of curvature $K \leq -1$ is a $CAT(-1)$ space. Hence \mathcal{R}_P has a \mathbb{Z}_2^m -invariant metric of curvature $K \leq -1$ if and only if \mathcal{U}_P has a W_P -invariant $CAT(-1)$ metric. The condition of the existence of a point o with a “standard” neighbourhood is technical and probably can be avoided.

The standard way to construct a metric on \mathcal{U}_P is as follows. The manifold \mathcal{U}_P has a natural cell decomposition with n -dimensional cells isomorphic to P . The dual cell decomposition \mathcal{U}_P^* is a cubical decomposition (see section 2). Now, choose an $a > 0$ and endow every cube of \mathcal{U}_P^* with the metric of a regular cube in \mathbb{H}^n of edge length a . Gromov [16] proved that the obtained metric is $CAT(-1)$ for some $a > 0$ if and only if a simple cell P satisfies condition (e) of Theorem 3.2, that is, satisfies the “no- Δ -condition” and the “no- \square -condition”. This $CAT(-1)$ metric on \mathcal{U}_P is W_P -invariant. Besides, it is *piecewise hyperbolic*, since every cell of \mathcal{U}_P^* is isometric to a cube in \mathbb{H}^n . Hence Theorem 3.2(e) immediately follows from Theorem 3.3.

Remark 3.4. The condition $K \leq -1$ in Theorem 3.3 cannot be replaced by a condition $K \leq 0$. Gromov [16] showed that the manifold \mathcal{U}_P has a W_P -invariant $CAT(0)$ metric whenever P is a flag simple cell, i. e., whenever K_P satisfies the “no- Δ -condition”. For example, the direct product $P_1 \times P_2$ of any flag simple cells is a flag simple cell. Hence the manifold $\mathcal{R}_{P_1 \times P_2}$ has a \mathbb{Z}_2^m -invariant metric of curvature $K \leq 0$. However, a result of Kotschick and Löh [17] implies that the manifold $\mathcal{R}_{P_1 \times P_2} = \mathcal{R}_{P_1} \times \mathcal{R}_{P_2}$ cannot be a URC-manifold if both cells P_1 and P_2 have positive dimensions. Nevertheless, the following conjecture seems to be reasonable.

Conjecture 3.5. *Suppose P is a simple cell that is not combinatorially equivalent to a direct product of two simple cells of positive dimensions. Then \mathcal{R}_P is a URC-manifold.*

Let us show that Theorem 3.3 implies Theorem 1.5. Indeed, let $W \subset \text{Isom}(\mathbb{H}^n)$ be a uniform right-angular reflection group and let P be a fundamental domain for W . Then $\mathcal{U}_P = \mathbb{H}^n$ has a W -invariant metric of constant curvature -1 . Theorem 3.3 yields that \mathcal{R}_P is a URC-manifold. Now, for a torsion-free subgroup $\Gamma \subset W$, the manifolds \mathcal{R}_P and \mathbb{H}^n/Γ possess a common finite-sheeted covering. Hence \mathbb{H}^n/Γ is a URC-manifold.

Remark 3.6. It is reasonable to ask whether the obtained URC-manifolds are smoothable. Indeed, if P is not only a simple cell, but a simple polytope in either \mathbb{R}^n or \mathbb{H}^n , then P has a natural structure of smooth manifold with corners. Hence the manifold \mathcal{R}_P has a \mathbb{Z}_2^m -equivariant smooth structure, see [7].

Remark 3.7. The proofs of Theorems 3.2 and 3.3 that will be given in the next two sections will never use that K_P is a combinatorial sphere. Actually, we might take for P a simple polyhedral complex dual to an arbitrary oriented pseudo-manifold K (see Remark 2.3). Then the analogues of Theorems 3.2 and 3.3 would assert that the pseudo-manifold \mathcal{R}_P satisfies the URC-condition, though we would not be able to guarantee that \mathcal{R}_P is a manifold.

4. A GROUP-THEORETIC CONSTRUCTION

Let Ω be a finite partially ordered set and let $m = |\Omega|$ be its cardinality. We shall write $\omega_1 \not\leq \omega_2$ to indicate that either $\omega_1 \geq \omega_2$ or ω_1 and ω_2 are incomparable. Let \mathbf{F} be the free product of m copies of the group \mathbb{Z}_2 indexed by the elements of the set Ω . The generator of the factor \mathbb{Z}_2 corresponding to an element $\omega \in \Omega$ will be denoted by x_ω . Thus,

$$\mathbf{F} = \langle x_\omega, \omega \in \Omega \mid x_\omega^2 = 1 \rangle.$$

For each $\omega \in \Omega$, we consider the automorphism $\psi_\omega \in \text{Aut}(\mathbf{F})$ such that

$$\psi_\omega(x_\gamma) = \begin{cases} x_\omega x_\gamma x_\omega & \text{if } \gamma < \omega, \\ x_\gamma & \text{if } \gamma \not\leq \omega \end{cases}$$

for every $\gamma \in \Omega$. We denote by Ψ the subgroup of $\text{Aut}(\mathbf{F})$ generated by the elements ψ_ω , $\omega \in \Omega$. Now, we consider the semi-direct product $\mathbf{F} \rtimes \Psi$ corresponding to the tautological action of $\Psi \subset \text{Aut}(\mathbf{F})$ on \mathbf{F} and we consider the elements $s_\omega = x_\omega \psi_\omega \in \mathbf{F} \rtimes \Psi$. Denote by S the set consisting of m elements s_ω and by W the subgroup of $\mathbf{F} \rtimes \Psi$ generated by the elements s_ω . Obviously, $\psi_\omega^2 = 1$ and $s_\omega^2 = 1$ for every $\omega \in \Omega$.

Proposition 4.1. *Suppose $\omega_1 < \omega_2$; then $\psi_{\omega_1} \psi_{\omega_2} = \psi_{\omega_2} \psi_{\omega_1}$ and $s_{\omega_1} s_{\omega_2} = s_{\omega_2} s_{\omega_1}$.*

Proof. Suppose $\gamma \in \Omega$. By a direct computation, we obtain that

$$\psi_{\omega_1}(\psi_{\omega_2}(x_\gamma)) = \psi_{\omega_2}(\psi_{\omega_1}(x_\gamma)) = \begin{cases} x_{\omega_2} x_{\omega_1} x_\gamma x_{\omega_1} x_{\omega_2} & \text{if } \gamma < \omega_1, \\ x_{\omega_2} x_\gamma x_{\omega_2} & \text{if } \gamma \not< \omega_1 \text{ and } \gamma < \omega_2, \\ x_\gamma & \text{if } \gamma \not< \omega_2. \end{cases}$$

Hence, the automorphisms ψ_{ω_1} and ψ_{ω_2} commute. Now, we have

$$\begin{aligned} s_{\omega_1} s_{\omega_2} &= x_{\omega_1} \psi_{\omega_1} x_{\omega_2} \psi_{\omega_2} = x_{\omega_1} x_{\omega_2} \psi_{\omega_1} \psi_{\omega_2}, \\ s_{\omega_2} s_{\omega_1} &= x_{\omega_2} \psi_{\omega_2} x_{\omega_1} \psi_{\omega_1} = x_{\omega_2} (x_{\omega_2} x_{\omega_1} x_{\omega_2}) \psi_{\omega_2} \psi_{\omega_1} = x_{\omega_1} x_{\omega_2} \psi_{\omega_1} \psi_{\omega_2}. \end{aligned}$$

Therefore, $s_{\omega_1} s_{\omega_2} = s_{\omega_2} s_{\omega_1}$. \square

Proposition 4.2. *(W, S) is a right-angular Coxeter system such that generators s_{ω_1} and s_{ω_2} commute if and only if ω_1 and ω_2 are comparable.*

Proof. We have already proved that the elements s_ω satisfy relations $s_\omega^2 = 1$ and $s_{\omega_1} s_{\omega_2} = s_{\omega_2} s_{\omega_1}$ whenever $\omega_1 < \omega_2$. We need to prove that these relations imply all relations among the elements s_ω . Let \underline{W} be the right-angular Coxeter group generated by the elements r_ω , $\omega \in \Omega$, with all relations among them following from the relations $r_\omega^2 = 1$ and $r_{\omega_1} r_{\omega_2} = r_{\omega_2} r_{\omega_1}$, $\omega_1 < \omega_2$. Then we have a well-defined homomorphism $i : \underline{W} \rightarrow \mathbf{F} \rtimes \Psi$ such that $i(r_\omega) = s_\omega$. We need to prove that i is injective. Consider the homomorphism $j : \mathbf{F} \rightarrow \underline{W}$ given by $j(x_\omega) = r_\omega$. Obviously, the homomorphism j is Ψ -invariant. Hence, there exists a well-defined homomorphism $j' : \mathbf{F} \rtimes \Psi \rightarrow \underline{W}$ such that $j'(x\psi) = j(x)$ for every $x \in \mathbf{F}$ and every $\psi \in \Psi$. Since the composite homomorphism

$$\underline{W} \xrightarrow{i} \mathbf{F} \rtimes \Psi \xrightarrow{j'} \underline{W}$$

is the identity isomorphism, we obtain that i is injective. \square

Remark 4.3. The above construction has the following generalization. Let \mathbb{F} be the free group with generators x_ω , $\omega \in \Omega$. Consider the automorphisms $\psi_\omega \in \text{Aut}(\mathbb{F})$ such that

$$\psi_\omega(x_\gamma) = \begin{cases} x_\omega x_\gamma x_\omega^{-1} & \text{if } \gamma < \omega, \\ x_\gamma & \text{if } \gamma \not< \omega \end{cases}$$

for every $\gamma \in \Omega$, and consider the elements $y_\omega = x_\omega^{-1} \psi_\omega \in \mathbb{F} \rtimes \text{Aut}(\mathbb{F})$. Then $y_{\omega_1} y_{\omega_2} = y_{\omega_2} y_{\omega_1}$ and these relations imply all relations among the elements y_ω . Therefore, the subgroup $A \subset \mathbb{F} \rtimes \text{Aut}(\mathbb{F})$ generated by the elements y_ω is the right-angular Artin group. The proof is similar to the proof of Proposition 4.2.

For $\omega \in \Omega$, let $\mathbf{F}_{>\omega} \subset \mathbf{F}$ be the subgroup generated by all elements $x_{\omega'}$ such that $\omega' > \omega$ and let $\mathbf{F}_{\geq\omega} \subset \mathbf{F}$ be the subgroup generated by all elements $x_{\omega'}$ such that $\omega' \geq \omega$. The definition of the automorphisms ψ_ω immediately implies

Proposition 4.4. *The subgroups $\mathbf{F}_{>\omega}$ and $\mathbf{F}_{\geq\omega}$ are Ψ -invariant. For each $\psi \in \Psi$, we have $\psi(x_\omega) = y x_\omega y^{-1}$ for a $y \in \mathbf{F}_{>\omega}$.*

Now, we consider the mapping $\mathbf{F} \rtimes \Psi \rightarrow \mathbf{F}$ such that $\psi x \mapsto x$ for every $x \in \mathbf{F}$ and every $\psi \in \Psi$, and we denote by θ the restriction of this mapping to W . The mapping $\theta : W \rightarrow \mathbf{F}$ is not a homomorphism, but it satisfies the following property.

Proposition 4.5. *Suppose $g \in W$ and $\omega \in \Omega$; then there exists $y \in \mathbf{F}_{>\omega}$ such that*

$$\theta(s_\omega g) = yx_\omega y^{-1}\theta(g).$$

Proof. Suppose $g = \psi x$, where $x \in \mathbf{F}$ and $\psi \in \Psi$; then $\theta(g) = x$. We have, $s_\omega g = \psi_\omega \psi x$, where $z = \psi^{-1}(x_\omega)$. Hence, $\theta(s_\omega g) = zx = z\theta(g)$. By Proposition 4.4, we have $z = yx_\omega y^{-1}$ for a $y \in \mathbf{F}_{>\omega}$. \square

Now, let H be a subgroup of finite index in \mathbf{F} . Denote by Ψ_H the subgroup of Ψ consisting of all automorphisms ψ such that $\psi(x)H = xH$ for every $x \in \mathbf{F}$. In particular, $\psi(H) = H$ for every $\psi \in \Psi_H$, hence, the semi-direct product $H \rtimes \Psi_H$ is well defined. It can be easily proved that the subgroup Ψ_H has finite index in Ψ . We put, $W_H = W \cap (H \rtimes \Psi_H)$. Then W_H is a subgroup of finite index in W .

Proposition 4.6. *Let g_1 and g_2 be elements of W such that $g_1 W_H = g_2 W_H$. Then $\theta(g_1)H = \theta(g_2)H$. Consequently, the mapping θ induces a well-defined mapping $\theta_H : W/W_H \rightarrow \mathbf{F}/H$.*

Proof. We have, $g_2 = g_1 g$ for some $g \in H \rtimes \Psi_H$. Suppose, $g_1 = \psi x$ and $g = \varphi h$, where $x \in \mathbf{F}$, $\psi \in \Psi$, $h \in H$, and $\varphi \in \Psi_H$. Then $\theta(g_1) = x$ and $\theta(g_2) = \varphi^{-1}(x)h$. Since $h \in H$ and $\varphi \in \Psi_H$, we see that $\theta(g_1)H = \theta(g_2)H$. \square

5. THE MANIFOLD M_{Π^n} IS URC

In this section we use the group-theoretic construction described in section 4 to prove Theorem 3.1. Since arcwise connected components of X can be treated separately, we may assume that X is arcwise connected.

Recall that a simplicial complex Z is called an n -dimensional *pseudo-manifold* if

- (1) each simplex of Z is contained in an n -dimensional simplex;
- (2) each $(n-1)$ -dimensional simplex of Z is contained in exactly two n -dimensional simplices.

All pseudo-manifolds under consideration are supposed to be compact and *strongly connected*. The latter means that any two n -dimensional simplices of Z can be connected by a sequence of n -dimensional simplices such that every two consecutive simplices have a common $(n-1)$ -dimensional face.

The following proposition follows easily from the definition of singular homology groups.

Proposition 5.1. *Let X be an arcwise connected topological space and let $z \in H_n(X, \mathbb{Z})$ be an arbitrary homology class. Then there exist a strongly connected oriented n -dimensional pseudo-manifold Z and a continuous mapping $f : Z \rightarrow X$ such that $f_*[Z] = z$.*

Hence, to prove that M_{Π^n} is a URC-manifold we suffice to show that for each strongly connected oriented n -dimensional pseudo-manifold Z , a multiple of the fundamental class of Z can be realised as an image of the fundamental class of a finite-sheeted covering of M_{Π^n} . We shall construct explicitly a finite-sheeted covering \widehat{M}_{Π^n} of M_{Π^n} and a continuous mapping $\widehat{M}_{\Pi^n} \rightarrow Z$ of non-zero degree.

Replacing Z with its barycentric subdivision, we may assume that vertices of Z admit a regular colouring in $n+1$ colours $1, \dots, n+1$. (This means that any two vertices of Z connected by an edge are of distinct colours.) For an n -dimensional simplex σ of Z , the

colouring of its vertices in colours $1, \dots, n+1$ provides the orientation of σ . We colour the simplex σ in either white or black colour depending on whether this orientation of σ coincides with the global orientation of Z or not. Obviously, any two simplices with a common facet have distinct colours. Thus we obtain a chess colouring of n -dimensional simplices of Z .

We denote by A the set of all n -dimensional simplices of Z and by A_+ and A_- the sets of all white and black n -dimensional simplices of Z respectively. For a simplex τ of Z , we denote by $A(\tau)$ the set of all n -dimensional simplices σ containing τ . We put $A_+(\tau) = A(\tau) \cap A_+$ and $A_-(\tau) = A(\tau) \cap A_-$.

Proposition 5.2. *For each simplex τ of Z such that $\dim \tau < n$, the number of white n -dimensional simplices σ containing τ is equal to the number of black n -dimensional simplices σ containing τ , i. e., $|A_+(\tau)| = |A_-(\tau)|$.*

Proof. Let $k = \dim \tau$. Each n -dimensional simplex $\sigma \in A(\tau)$ contains exactly $n - k$ simplices ρ such that $\dim \rho = n - 1$ and $\rho \supset \tau$. On the other hand, every $(n - 1)$ -dimensional simplex ρ containing τ is contained in exactly one simplex belonging to $A_+(\tau)$ and in exactly one simplex belonging to $A_-(\tau)$. Hence, the number of $(n - 1)$ -dimensional simplices ρ containing τ is equal to both $(n - k)|A_+(\tau)|$ and $(n - k)|A_-(\tau)|$. Therefore, $|A_+(\tau)| = |A_-(\tau)|$. \square

This proposition implies that we can pair off elements of the sets $A_+(\tau)$ and $A_-(\tau)$. For every τ , we choose arbitrarily such pairing λ_τ . Then $\lambda_\tau : A(\tau) \rightarrow A(\tau)$ is a bijection such that $\lambda_\tau(A_\pm(\tau)) = A_\mp(\tau)$ and $\lambda_\tau^2 = \text{id}$.

Consider a non-empty proper subset $\omega \subset [n+1]$. Denote by \mathcal{T}_ω the set of all simplices τ of Z such that the set of colours of vertices of τ coincides with ω . If $|\omega| = k$, then the set \mathcal{T}_ω consists of $(k - 1)$ -dimensional simplices. Obviously, each simplex $\sigma \in A$ contains exactly one simplex $\tau \in \mathcal{T}_\omega$, which will be called the *face of σ of type ω* . Hence the set A is the disjoint union of the sets $A(\tau)$, $\tau \in \mathcal{T}_\omega$. Let $\lambda_\omega : A \rightarrow A$ be the bijection whose restriction to every set $A(\tau)$ coincides with λ_τ . Then λ_ω is a permutation of the set A satisfying the following conditions:

- (1) $\lambda_\omega(A_+) = A_-$ and $\lambda_\omega(A_-) = A_+$.
- (2) $\lambda_\omega^2 = 1$.
- (3) For each $\sigma \in A$, the simplices σ and $\lambda_\omega(\sigma)$ have common face of type ω .

Let $\Delta^n \subset \mathbb{R}^{n+1}$ be the standard simplex with vertices

$$e_1 = (1, 0, \dots, 0), \quad e_2 = (0, 1, \dots, 0), \quad \dots, \quad e_{n+1} = (0, 0, \dots, 1).$$

For a non-empty proper subset $\omega \subset [n+1]$, we denote by Δ_ω the face of Δ^n with vertices e_i such that $i \in \omega$. For each simplex $\sigma \in A$, let $\iota_\sigma : \Delta^n \rightarrow \sigma$ be the linear isomorphism taking every vertex e_i to the vertex of σ of colour i . Obviously, ι_σ preserves the orientation if $\sigma \in A_+$ and reverses the orientation if $\sigma \in A_-$. We shall denote the point $\iota_\sigma(p)$ by $[p, \sigma]$. Then

$$[p, \lambda_\omega(\sigma)] = [p, \sigma] \tag{1}$$

whenever $p \in \Delta_\omega$.

Now, let Ω be the set of all non-empty proper subsets $\omega \subset [n+1]$ partially ordered by inclusion; then $m = |\Omega| = 2^{n+1} - 2$. We consider the construction in section 4 for this Ω . Recall that facets F_ω of Π^n are in one-to-one correspondence with subsets $\omega \in \Omega$, and $F_{\omega_1} \cap F_{\omega_2} \neq \emptyset$ if and only if either $\omega_1 \subset \omega_2$ or $\omega_2 \subset \omega_1$. Hence the right-angular Coxeter group W corresponding to the partially ordered set Ω coincides with the right-angular Coxeter group W_{Π^n} . We have $M_{\Pi^n} = \mathcal{U}_{\Pi^n} / \Gamma$, where Γ is the kernel of the homomorphism $\rho : W \rightarrow \mathbb{Z}_2^n$ such that $\rho(s_\omega) = b_{|\omega|}$, $\omega \in \Omega$.

Let us construct a degree 1 mapping $\pi : \Pi^n \rightarrow \Delta^n$ such that $\pi(F_\omega) \subset \Delta_\omega$ for every $\omega \in \Omega$. Map the barycentre of Π^n to the barycentre of Δ^n . Each face F of Π^n can be uniquely written as $F_{\omega_1} \cap \dots \cap F_{\omega_k}$, $\omega_1 \subset \dots \subset \omega_k$. Map the barycentre of F to the barycentre of Δ_{ω_1} . Now the mapping π is defined on the vertices of the barycentric subdivision of Π^n . Extend this mapping linearly to every simplex of the barycentric subdivision of Π^n .

Denote by S_A the group of permutations of the set A . Consider the homomorphism $\Lambda : \mathbf{F} \rightarrow S_A$ given by $\Lambda(x_\omega) = \lambda_\omega$. This homomorphism yields the action of the group \mathbf{F} on the set A . Since Z is strongly connected, this action is transitive. We shall write $g \cdot \sigma = \Lambda(g)(\sigma)$ for $g \in \mathbf{F}$, $\sigma \in A$. Choose a simplex $\sigma_0 \in A_+$ and define the mapping $f : \mathcal{U}_{\Pi^n} \rightarrow Z$ by

$$f([p, g]) = [\pi(p), \theta(g) \cdot \sigma_0].$$

Let $H \subset \mathbf{F}$ be the stabilizer of σ_0 . The subgroup H has finite index, since the set A is finite. Then W_H is a subgroup of finite index in W . Let $\Gamma_H = W_H \cap \Gamma$ and $\widehat{M}_{\Pi^n} = \mathcal{U}_{\Pi^n} / \Gamma_H$.

Proposition 5.3. *The mapping f is well defined and invariant under the right action of W_H on \mathcal{U}_{Π^n} . Hence f induces a well-defined mapping $f_1 : \widehat{M}_{\Pi^n} \rightarrow Z$. The degree of f_1 is equal to $\frac{|W : \Gamma_H|}{|A|}$.*

Proof. To prove that f is well defined we need to show that

$$[\pi(p), \theta(g) \cdot \sigma_0] = [\pi(p), \theta(g') \cdot \sigma_0]$$

whenever $g'g^{-1} \in W(p)$. Let $F_{\omega_1} \cap \dots \cap F_{\omega_k}$ be the minimal face of Π^n that contains p . We may renumerate the subsets ω_i so that $\omega_1 \subset \dots \subset \omega_k$. Then $\pi(p) \in \Delta_{\omega_1}$. The subgroup $W(p)$ is generated by $s_{\omega_1}, \dots, s_{\omega_k}$. Since $g'g^{-1} \in W(p)$, Proposition 4.5 implies that $\theta(g') = x\theta(g)$ for an $x \in \mathbf{F}_{\geq \omega_1}$. Then $\Lambda(x)$ belongs to the subgroup of S_A generated by all λ_ω such that $\omega \supseteq \omega_1$. Let $\sigma = \theta(g) \cdot \sigma_0$; then $\theta(g') \cdot \sigma_0 = x \cdot \sigma$. Equation (1) implies that $[\pi(p), x \cdot \sigma] = [\pi(p), \sigma]$. Therefore f is well defined. Proposition 4.6 immediately yields that f is W_H -invariant.

Let $\Gamma_H = W_H \cap \Gamma$. The manifold \mathcal{U}_{Π^n} is glued of permutahedra indexed by elements $g \in W$. We denote by Π_g the permutahedron corresponding to g . The restriction $f|_{\Pi_g}$ coincides with the composite mapping

$$\Pi_g \xrightarrow{\iota_g^{-1}} \Pi^n \xrightarrow{\pi} \Delta^n \xrightarrow{\iota_{\theta(g) \cdot \sigma_0}} \theta(g) \cdot \sigma_0,$$

where $\iota_g : \Pi^n \rightarrow \mathcal{U}_{\Pi^n}$ is the mapping given by $\iota_g(p) = [p, g]$. All mappings in this diagram have degrees ± 1 . The mapping π has degree 1. The mapping ι_g has degree 1 if and only if the element g is represented by a word of even length in generators s_ω . The mapping $\iota_{\theta(g) \cdot \sigma_0}$ has degree 1 if and only if $\theta(g) \cdot \sigma_0 \in A_+$. Obviously, $\theta(g) \cdot \sigma_0 \in A_+$ if and only if the element $\theta(g) \in \mathbf{F}$ is represented by a word of even length in generators x_ω . Proposition 4.5 implies that the parity of the length of the word in x_ω representing $\theta(g)$ coincides with the parity of the length of the word in s_ω representing g . Therefore, the mapping $f|_{\Pi_g} : \Pi_g \rightarrow \theta(g) \cdot \sigma_0$ always has degree 1. Now we see that \widehat{M}_{Π^n} is a connected oriented manifold glued out of $|W : \Gamma_H|$ permutahedra and Z is a strongly connected oriented pseudo-manifold glued out of $|A|$ simplices. Each cell of \widehat{M}_{Π^n} is mapped by f_1 onto a simplex of Z with degree 1. Hence, the degree of f_1 is equal to $\frac{|W : \Gamma_H|}{|A|}$. \square

Remark 5.4. By a theorem of Thom [23], for each n , there is a positive integer $k(n)$ such that the class $k(n)z$ is realisable for every X and every $z \in H_n(X, \mathbb{Z})$. Estimates for $k(n)$ were obtained by Novikov [20] and Buchstaber [3]. The best known estimate is that $k(n)$ divides the number $\prod p^{\lfloor \frac{n-2}{2(p-1)} \rfloor}$, where the product is taken over all odd primes p ,

see [3]. Unfortunately, the combinatorial construction described above does not allow us to obtain a reasonable estimate for $k(n)$. The multiplicity $\frac{|W:\Gamma_H|}{|A|}$ obtained by this construction can be huge.

6. PROOF OF THEOREM 3.2

In this section we shall prove all assertions of Theorem 3.2 except for (e). It was shown in section 3 that Theorem 3.2(e) follows from Theorem 3.3. The proof of Theorem 3.3 will be given in the next section.

Since \mathcal{R}_{Π^n} is a URC-manifold and K_{Π^n} is isomorphic to $(\partial\Delta^n)'$, the following proposition implies Theorem 3.2(a).

Proposition 6.1. *Let P_1 and P_2 be n -dimensional simple cells such that there exists a simplicial mapping $\varphi : K_{P_1} \rightarrow K_{P_2}$ of non-zero degree. Then $\mathcal{R}_{P_1} \geq \mathcal{R}_{P_2}$.*

Proof. Let F_1, \dots, F_{m_1} be the facets of P_1 and let G_1, \dots, G_{m_2} be the facets of P_2 . Let $\bar{\varphi} : P_1 \rightarrow P_2$ be the mapping induced by φ . Identifying the vertex sets of K_{P_1} and K_{P_2} with the sets $[m_1]$ and $[m_2]$ respectively, we may regard φ as a mapping $[m_1] \rightarrow [m_2]$. Let $\mu : \mathbb{Z}_2^{m_1} \rightarrow \mathbb{Z}_2^{m_2}$ be the homomorphism given by $\mu(a_i) = b_{\varphi(i)}$, where a_1, \dots, a_{m_1} and b_1, \dots, b_{m_2} are the standard generators of the groups $\mathbb{Z}_2^{m_1}$ and $\mathbb{Z}_2^{m_2}$ respectively. Since $\deg \varphi \neq 0$, we easily see that μ is surjective. Now we consider the mapping $f : \mathcal{R}_{P_1} \rightarrow \mathcal{R}_{P_2}$ given by $f([p, g]) = [\bar{\varphi}(p), \mu(g)]$. Since $\bar{\varphi}(F_i) \subset G_{\varphi(i)}$ for every i , we see that the mapping f is well defined. Obviously, f maps each n -cell of \mathcal{R}_{P_1} onto an n -cell of \mathcal{R}_{P_2} with degree $\deg \varphi$. Therefore, $\deg f = 2^{m_2-m_1} \deg \varphi \neq 0$. \square

To prove Theorems 3.2(b), (c), (d) we shall show that (b) \Rightarrow (a) and (d) \Rightarrow (c) \Rightarrow (a).

Proof of (b) \Rightarrow (a). For any $(n-1)$ -dimensional combinatorial sphere L , there exists a simplicial mapping $\varphi : L \rightarrow \partial\Delta^n$ of degree 1. This mapping induces a required simplicial mapping $\varphi' : K_P = L' \rightarrow (\partial\Delta^n)'$. \square

Let P be a flag simple cell and let $0 < \varepsilon < \pi$. We say that P is ε -sparse if there exists a homeomorphism $h : \partial P \rightarrow \mathbb{S}^{n-1}$ such that $\text{dist}(h(F_1), h(F_2)) \geq \varepsilon$ for any two non-intersecting facets F_1 and F_2 of P , and $\text{diam}(h(F)) \leq \pi - \varepsilon$ for every facet F of P . (The latter condition is technical. It guarantees that the open $\frac{\varepsilon}{2}$ -neighborhood of every set $h(F)$ does not contain antipodal points.)

Proposition 6.2. *Let P_1 and P_2 be n -dimensional simple cells such that P_1 is ε -fine and P_2 is flag and ε -sparse. Then there exists a simplicial mapping $K_{P_1} \rightarrow K_{P_2}$ of non-zero degree.*

Proof. Let F_1, \dots, F_{m_1} be the facets of P_1 and let G_1, \dots, G_{m_2} be the facets of P_2 . Then $K_1 = K_{P_1}$ and $K_2 = K_{P_2}$ are simplicial complexes on the vertex sets $\{u_1, \dots, u_{m_1}\}$ and $\{v_1, \dots, v_{m_2}\}$ respectively. Let $f : K_1 \rightarrow \mathbb{S}^{n-1}$ be the mapping of non-zero degree such that $\text{diam}(f(\sigma)) < \varepsilon$ for every simplex σ of K_1 . Let $h : \partial P_2 \rightarrow \mathbb{S}^{n-1}$ be the homeomorphism such that $\text{dist}(h(G_i), h(G_j)) \geq \varepsilon$ whenever $G_i \cap G_j = \emptyset$, and $\text{diam}(h(G_i)) \leq \pi - \varepsilon$ for every i . We define a mapping $\varphi : [m_1] \rightarrow [m_2]$ in the following way. For every $i \in [m_1]$, consider the point $h^{-1}(f(u_i)) \in \partial P_2$, choose an arbitrary facet G_j containing this point, and put $\varphi(i) = j$. Now let us prove that the mapping φ induces a simplicial mapping $\Phi : K_1 \rightarrow K_2$. For every i , we put $\Phi(u_i) = v_{\varphi(i)}$. Suppose $[u_{i_1} \dots u_{i_k}]$ is a simplex of K_1 . Then $\text{dist}(f(u_{i_p}), f(u_{i_q})) < \varepsilon$ for any p and q . But $f(u_{i_p}) \in h(G_{\varphi(i_p)})$ and $f(u_{i_q}) \in h(G_{\varphi(i_q)})$. Hence, $G_{\varphi(i_p)} \cap G_{\varphi(i_q)} \neq \emptyset$. Since K_2 is a flag complex, we obtain that $G_{\varphi(i_1)} \cap \dots \cap G_{\varphi(i_k)} \neq \emptyset$. Therefore, the vertices $\Phi(u_{i_1}), \dots, \Phi(u_{i_k})$ span a simplex in K_2 . Thus, Φ uniquely extends to a simplicial mapping $\Phi : K_1 \rightarrow K_2$.

To show that Φ has non-zero degree, we need to prove that the mappings $\Phi : K_1 \rightarrow K_2$ and $h^{-1}f : K_1 \rightarrow \partial P_2 = K_2$ are homotopic. Equivalently, we need to prove that the mappings $h\Phi$ and f of K_1 to \mathbb{S}^{n-1} are homotopic. Let $x \in K_1 = \partial P_1$ be an arbitrary point. Let F_i be a facet of P_1 containing x and let σ be a simplex of K_1 containing x . Then $u_i \in \sigma$, hence, $\text{dist}(f(x), f(u_i)) < \varepsilon$. Since $\Phi(F_i) \subset G_{\varphi(i)}$, we obtain that $h(\Phi(x)) \in h(G_{\varphi(i)})$. On the other hand, we have $f(u_i) \in h(G_{\varphi(i)})$ by the definition of φ . Hence, $\text{dist}(h(\Phi(x)), f(u_i)) \leq \text{diam}(h(G_{\varphi(i)})) \leq \pi - \varepsilon$. Therefore, $\text{dist}(h(\Phi(x)), f(x)) < \pi$. Since this inequality holds for every $x \in K_1$, we see that $h\Phi$ is homotopic to f . \square

Corollary 6.3. *Let P_1 and P_2 be n -dimensional simple cells such that P_1 is ε -fine and P_2 is flag and ε -sparse. Then $\mathcal{R}_{P_1} \geq \mathcal{R}_{P_2}$.*

Proposition 6.4. *For every $n \geq 2$, the permutahedron Π^n is ε_n -sparse.*

Proof. We consider the standard realisation of Π^n in \mathbb{R}^{n+1} whose vertices are obtained by permuting the coordinates of the point $(1, 2, \dots, n+1)$. Let $o = (\frac{n+2}{2}, \dots, \frac{n+2}{2})$ be the centre of Π^n . It is easy to compute that the circumscribed sphere of Π^n has radius $R_n = \sqrt{\frac{n(n+1)(n+2)}{12}}$. Identify \mathbb{S}^{n-1} with the unit sphere with centre at o in the hyperplane containing Π^n , and let $h : \partial \Pi^n \rightarrow \mathbb{S}^{n-1}$ be the radial projection from o . Suppose, x and y are points belonging to non-intersecting facets F_{ω_1} and F_{ω_2} of Π^n and let $\xi = (\xi_1, \dots, \xi_{n+1})$ and $\eta = (\eta_1, \dots, \eta_{n+1})$ be the vectors \vec{ox} and \vec{oy} respectively. Since $F_{\omega_1} \cap F_{\omega_2} = \emptyset$, we see that neither $\omega_1 \subset \omega_2$ nor $\omega_2 \subset \omega_1$. Choose any $j \in \omega_1 \setminus \omega_2$ and any $k \in \omega_2 \setminus \omega_1$. Then $\xi_j - \xi_k \geq 1$ and $\eta_k - \eta_j \geq 1$. We have,

$$\cos \text{dist}(h(x), h(y)) = \frac{(\xi, \eta)}{|\xi||\eta|} = \frac{\sum_{i \neq j, k} \xi_i \eta_i + \xi_j \eta_k + \xi_k \eta_j - (\xi_j - \xi_k)(\eta_k - \eta_j)}{|\xi||\eta|} \leq 1 - \frac{1}{R_n^2}$$

Hence, $\text{dist}(h(x), h(y)) \geq \varepsilon_n$.

Now, we need to prove that $\text{diam}(h(F_\omega)) \leq \pi - \varepsilon_n$ for every ω . Let B be the circumscribed ball of Π^n and let H_ω be the affine plane spanned by F_ω . It is easy to see that the distance from o to H_ω is at least $r_n = \frac{1}{2}\sqrt{n(n+1)}$. (The value r_n is attained if $|\omega|$ is either 1 or n .) Since $F_\omega \subset H_\omega \cap B$, we have

$$\text{diam}(h(F_\omega)) \leq 2 \arccos \frac{r_n}{R_n} = \pi - \arccos \left(1 - \frac{6}{n+2} \right) \leq \varepsilon_n.$$

Thus Π^n is ε_n -sparse. \square

Propositions 6.2 and 6.4 imply that (c) \Rightarrow (a). Since

$$\rho_n = \text{arsinh} \cot \frac{\varepsilon_n}{2},$$

the following proposition yields (d) \Rightarrow (c).

Proposition 6.5. *Let $0 < \varepsilon \leq \frac{\pi}{2}$ and let $P \subset \mathbb{H}^n$ be a simple convex polytope whose interior contains a closed ball of radius $\rho = \text{arsinh} \cot \frac{\varepsilon}{2}$. Then P is ε -fine.*

Proof. Let $B \subset P$ be a ball of radius ρ and let o be its centre. Identify \mathbb{S}^{n-1} with the sphere of unit vectors in $T_o \mathbb{H}^n$. Define the mapping $f : \mathbb{H}^n \setminus \{o\} \rightarrow \mathbb{S}^{n-1}$ by taking every point x to the tangent vector to the line segment $[ox]$ at o . Obviously, the degree of $f|_{\partial P}$ is equal to 1.

Let F_1, \dots, F_m be the facets of P and let v_1, \dots, v_m be the corresponding vertices of K_P . For each i , let p_i be the orthogonal projection of the point o onto the hyperplane spanned

by F_i and let $\xi_i = f(p_i)$. For each point $x \in F_i$, $op_i x$ is a triangle with right angle at the vertex p_i and angle $\text{dist}_{\mathbb{S}^{n-1}}(f(x), \xi_i)$ at the vertex o . Hence,

$$\cot(\text{dist}_{\mathbb{S}^{n-1}}(f(x), \xi_i)) = \sinh(\text{dist}_{\mathbb{H}^n}(o, p_i)) \coth(\text{dist}_{\mathbb{H}^n}(x, p_i)) > \sinh \rho = \cot \frac{\varepsilon}{2}$$

Therefore,

$$\text{dist}_{\mathbb{S}^{n-1}}(f(x), \xi_i) < \frac{\varepsilon}{2} \quad (2)$$

Consequently, $\text{dist}_{\mathbb{S}^{n-1}}(\xi_i, \xi_j) < \varepsilon$ whenever $F_i \cap F_j \neq \emptyset$.

Define a mapping $\tilde{f} : K_P \rightarrow \mathbb{S}^{n-1}$ by putting

$$\tilde{f}(x) = \frac{\beta_1 \xi_{i_1} + \dots + \beta_k \xi_{i_k}}{|\beta_1 \xi_{i_1} + \dots + \beta_k \xi_{i_k}|}$$

for a point x that has barycentric coordinates β_1, \dots, β_k in a simplex $[v_{i_1} \dots v_{i_k}]$. Obviously, \tilde{f} is well defined. We have $\tilde{f}(v_i) = \xi_i$. For each simplex $\Delta = [v_{i_1} \dots v_{i_k}]$ of K_P , $\tilde{f}(\Delta)$ is the convex hull of the points $\xi_{i_1}, \dots, \xi_{i_k}$ in \mathbb{S}^{n-1} . Pairwise distances between the points $\xi_{i_1}, \dots, \xi_{i_k}$ are less than ε . Since $\varepsilon \leq \frac{\pi}{2}$, this implies that $\text{diam}(\tilde{f}(\Delta)) < \varepsilon$. Besides, inequality (2) implies that the set $f(\Delta)$ is contained in an $\frac{\varepsilon}{2}$ -neighbourhood of the convex simplex $\tilde{f}(\Delta)$. It follows that $\text{dist}_{\mathbb{S}^{n-1}}(\tilde{f}(x), f(x)) < \pi$ for every $x \in K_P = \partial P$. Hence \tilde{f} is homotopic to $f|_{\partial P}$. Therefore, \tilde{f} has degree 1. Thus P is ε -fine. \square

7. PROOFS OF THEOREMS 1.5 AND 3.3

We have shown in section 3 that Theorem 3.3 implies Theorem 1.5. Nevertheless, it is useful to give a proof of Theorem 1.5 first, because this proof contains all ideas behind the proof of Theorem 3.3.

Proof of Theorem 1.5. Let \mathcal{M} be the set of all mirrors of reflections $s \in W$. Choose a point $o \in \mathbb{H}^n$ and denote by $\mathcal{M}_{>\rho_n}$ the set of all mirrors $H \in \mathcal{M}$ such that the distance from o to H is greater than ρ_n . For each $H \in \mathcal{M}_{>\rho_n}$, let $\Lambda_H^+ \subset \mathbb{H}^n$ be the closed half-space bounded by H and containing o . Denote by P_1 the intersection of all half-spaces Λ_H^+ , $H \in \mathcal{M}_{>\rho_n}$, see Figure 1. (In this figure the central pentagon is P , the dashed circle is the sphere of centre o and radius ρ_n , and the polygon bounded by the thick line is P_1 .) Obviously, P_1 is a compact convex polytope. Since mirrors in \mathcal{M} are orthogonal whenever intersect, we see that the polytope P_1 is right-angular. In particular, it is simple. Let W_1 be the group generated by the reflections in facets of P_1 . Then W_1 is a subgroup of finite index in W . Let F_1, \dots, F_{m_1} be the facets of P_1 and let s_1, \dots, s_{m_1} be the orthogonal reflections in these facets respectively. Consider the homomorphism $\eta_1 : W_1 \rightarrow \mathbb{Z}_2^{m_1}$ that takes each s_i to the generator of the i th factor \mathbb{Z}_2 and put $\Gamma_1 = \ker \eta_1$. Then Γ_1 acts freely on \mathbb{H}^n and the quotient \mathbb{H}^n/Γ_1 is homeomorphic to \mathcal{R}_{P_1} . Since the interior of P_1 contains a ball of radius ρ_n , we obtain that \mathbb{H}^n/Γ_1 is a URC-manifold. But the manifolds \mathbb{H}^n/Γ and \mathbb{H}^n/Γ_1 possess a common finite-sheeted covering, since the groups Γ and Γ_1 are commensurable. Therefore, \mathbb{H}^n/Γ is a URC-manifold. \square

Proof of Theorem 3.3. The idea is to mimic the above proofs of Proposition 6.5 and Theorem 1.5 using the $CAT(-1)$ inequality instead of constant negative curvature to obtain an analogue of inequality (2). However, we shall face some additional difficulties on this way.

The group $W = W_P$ acts by reflections on the $CAT(-1)$ manifold \mathcal{U}_P . Let \mathcal{M} be the set of all mirrors of reflections $s \in W$. Let $o \in \mathcal{U}_P$ be a point with a neighbourhood isometric to an open subset of \mathbb{H}^n . Then the tangent space $T_o \mathcal{U}_P$ is well defined. Let ρ be

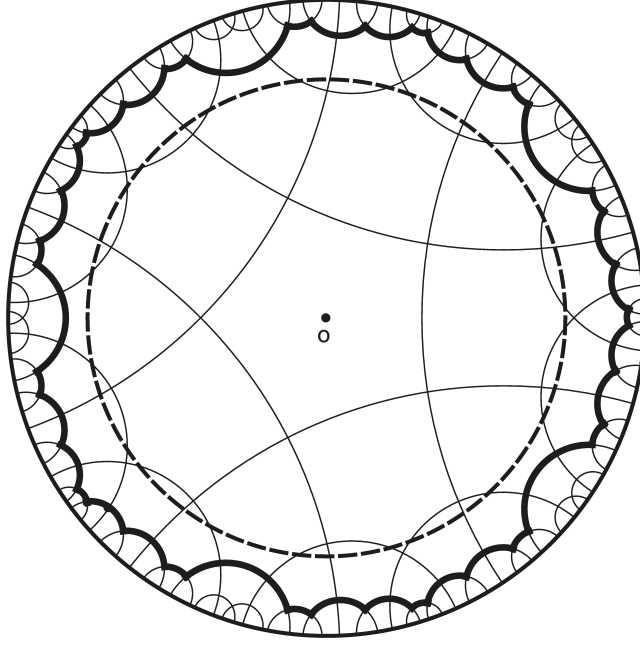


FIGURE 1. Polytope P_1

a sufficiently large positive number. (A particular ρ will be chosen later.) Denote by $\mathcal{M}_{>\rho}$ the set of all mirrors $H \in \mathcal{M}$ such that the distance from o to H is greater than ρ . For each $H \in \mathcal{M}_{>\rho}$, let $\Lambda_H^+ \subset \mathcal{U}_P$ be the closed half-space bounded by H and containing o . Denote by P_1 the intersection of all half-spaces Λ_H^+ , $H \in \mathcal{M}_{>\rho}$. Let $H_1, \dots, H_{m_1} \in \mathcal{M}_{>\rho}$ be all mirrors intersecting P_1 ; then the subsets $F_i = H_i \cap P_1$ will be called *facets* of P_1 and their non-empty intersections will be called *faces* of P_1 . (Since W is a *right-angled* Coxeter group, it follows that every F_i is $(n-1)$ -dimensional.) Let $W_1 \subset W$ be the subgroup of finite index generated by the reflections in mirrors H_1, \dots, H_{m_1} .

If P_1 were a simple cell, \mathcal{R}_{P_1} would be a finite-sheeted covering of \mathcal{R}_P . Hence we would suffice to prove that \mathcal{R}_{P_1} is a URC-manifold. The author does not know whether P_1 is always a simple cell (see Remark 7.3 below). Nevertheless, P_1 “looks like” a simple cell so that we can mimic the proofs of Propositions 6.1, 6.2, and 6.5 in this context. The main difficulty is that we cannot identify P_1 with a cone over the dual combinatorial sphere K_{P_1} and, hence, should avoid usage of K_{P_1} . We start with the following obvious

Proposition 7.1. *There exists a mapping $t : \Pi^n \rightarrow \Pi^n$ such that $t(U_\omega) \subset F_\omega$ for some open neighbourhoods U_ω of facets $F_\omega \subset \Pi^n$.*

Notice that we denote facets of P_1 by F_i and facets of Π^n by F_ω . This will not lead to a confusion.

Obviously, the mapping t has degree 1. Now we consider the mapping

$$\varphi = th^{-1} : \mathbb{S}^{n-1} \rightarrow \partial\Pi^n,$$

where $h : \partial\Pi^n \rightarrow \mathbb{S}^{n-1}$ is the radial projection constructed in the proof of Proposition 6.4. Since the permutahedron has finitely many facets and every facet is compact, we obtain that the mapping φ satisfies the following property:

(*) *There is an $\varepsilon > 0$ such that $\varphi(x) \in F_\omega$ whenever $\text{dist}(x, h(F_\omega)) \leq \varepsilon$.*

Now we are ready to choose a particular $\rho > 0$. We take $\rho = \text{arsinh} \cot \frac{\varepsilon}{2}$.

Identify \mathbb{S}^{n-1} with the unit sphere in $T_o\mathcal{U}_P$. Since \mathcal{U}_P is a $CAT(-1)$ manifold, the point o is joined with every point $x \in \mathcal{U}_P$ by a unique geodesic segment. We denote by $f(x)$ the unit tangent vector to this geodesic segment at o . Then $f : \mathcal{U}_P \setminus \{o\} \rightarrow \mathbb{S}^{n-1}$ is a continuous mapping.

Proposition 7.2. *For each facet F_i of P_1 , the diameter of $f(F_i)$ does not exceed ε .*

Proof. Let $H_i \in \mathcal{M}_{>\rho}$ be the mirror such that $H_i \cap P_1 = F_i$. Then the distance d from o to H_i is greater than ρ . Since H_i is closed and the balls in \mathcal{U}_P are compact, we obtain that there exists a point $p \in H_i$ such that $\text{dist}(o, p) = d$. Let x be a point in F_i . Consider the comparison triangle $o^*x^*p^*$ in \mathbb{H}^2 for the triangle oxp . The geodesic segment $[xp]$ is contained in H_i . Hence, the distance from o to every point y of $[xp]$ is greater than or equal to d . By $CAT(-1)$ inequality, the distance from o^* to every point of the segment $[x^*p^*]$ is greater than or equal to d . But the length of the segment $[o^*p^*]$ is equal to d . Hence the angle β^* of the triangle $o^*x^*p^*$ at the vertex p^* is greater than or equal to $\frac{\pi}{2}$. Let α^* be the angle of the triangle $o^*x^*p^*$ at the vertex o^* . Then

$$\cot \alpha^* \geq \sinh d > \sinh \rho = \cot \frac{\varepsilon}{2}.$$

Therefore, $\alpha^* < \frac{\varepsilon}{2}$.

A neighbourhood of o in \mathcal{U}_P is isometric to an open subset of \mathbb{H}^n . Hence the angle α of the triangle oxp at the vertex o is well defined and is equal to $\text{dist}(f(x), f(p))$. The $CAT(-1)$ inequality easily implies that

$$\text{dist}(f(x), f(p)) = \alpha \leq \alpha^* < \frac{\varepsilon}{2}.$$

Hence, for any two points $x, y \in F_i$, we have $\text{dist}(f(x), f(y)) < \varepsilon$. \square

For each $i \in [m_1]$, choose a subset ω_i such that $f(F_i) \cap h(F_{\omega_i}) \neq \emptyset$. Since $\text{diam}(f(F_i)) < \varepsilon$, property (*) implies that $\varphi(f(F_i)) \subset F_{\omega_i}$. Define the homomorphism $\xi : \mathbb{Z}_2^{m_1} \rightarrow \mathbb{Z}_2^n$ by $\xi(a_i) = b_{|\omega_i|}$, where a_1, \dots, a_{m_1} are the generators of $\mathbb{Z}_2^{m_1}$ and b_1, \dots, b_n are the generators of \mathbb{Z}_2^n .

Though we are not sure that P_1 is a simple cell, we can apply the construction of \mathcal{R}_{P_1} described in section 2. Namely, we put

$$\mathcal{R}_{P_1} = (P_1 \times \mathbb{Z}_2^{m_1}) / \sim,$$

where $(p, g) \sim (p', g')$ if and only if $p = p'$ and $g'g^{-1}$ belongs to the subgroup of $\mathbb{Z}_2^{m_1}$ generated by all a_i such that $p \in F_i$. Let $\eta_1 : W_1 \rightarrow \mathbb{Z}_2^{m_1}$ be the homomorphism that takes the reflection in every mirror H_i to a_i , and let $\Gamma_1 = \ker \eta_1$. Then Γ_1 is torsion-free and $\mathcal{R}_{P_1} = \mathcal{U}_P / \Gamma_1$. Therefore \mathcal{R}_{P_1} is a PL manifold.

Now we define a mapping $\Phi : \mathcal{R}_{P_1} \rightarrow M_{\Pi^n}$ by

$$\Phi([p, g]) = [\varphi(f(p)), \xi(g)].$$

Since $\varphi(f(F_i)) \subset F_{\omega_i}$, we obtain that Φ is well defined. Since f and φ are degree 1 mappings, we obtain that the degree of Φ is equal to $2^{m_1-n} \neq 0$. Therefore, \mathcal{R}_{P_1} is a URC-manifold. Since \mathcal{R}_P and \mathcal{R}_{P_1} possess a common finite-sheeted covering, we obtain that \mathcal{R}_P is a URC-manifold. \square

Remark 7.3. The set P_1 is a convex subset of a $CAT(-1)$ manifold, hence, P_1 is contractible. Similarly, faces of P_1 are convex and, hence, contractible. Stone [22] proved that closed metric balls in a piecewise Euclidean or piecewise hyperbolic $CAT(0)$ -manifold are homeomorphic to the standard ball, see also [11] and [9]. This proof seems to be possible to extend to the case of an arbitrary convex subset. However, to guarantee that

P_1 is a simple cell we still need a stronger result. Namely, we need to prove that the homeomorphisms of facets of P_1 onto the standard balls can be chosen piecewise linear and compatible to each other. The author does not know whether this is possible or not. Actually, even if it were possible to deduce that P_1 is a simple cell from results of [22], [11], and [9], we would prefer to avoid usage of these results, since they are based on hard theorems of geometric topology.

REFERENCES

- [1] M. Bridson, A. Haefliger, *Metric Spaces of Nonpositive Curvature*, Springer, New York, 1999.
- [2] R. Brooks, *On branched coverings of 3-manifolds which fiber over the circle*, J. Reine Angew. Math. **362** (1985), 87–101.
- [3] V. M. Buchstaber, *Modules of differentials of the Atiyah-Hirzebruch spectral sequence, I, II*, Matem. Sb. **78(120)**:2, 307–320; **83(125)**:1(9), 61–76; English transl., Math. USSR Sb. **7**:2 (1969), 299–313; **12**:1 (1970), 59–75.
- [4] V. M. Buchstaber, T. E. Panov, *Torus actions and their applications in topology and combinatorics*, University Lecture Series, no. 24, Amer. Math. Soc., Providence, RI, 2002.
- [5] J. A. Carlson, D. Toledo, *Harmonic mappings of Kähler manifolds to locally symmetric spaces*, Publ. Math. I.H.E.S. **69** (1989), 173–201.
- [6] R. M. Charney, M. W. Davis, *Strict hyperbolization*, Topology **34** (1995), 329–350.
- [7] M. W. Davis, *Groups generated by reflections and aspherical manifolds not covered by Euclidean space*, Ann. Math. (2) **117**:2 (1983), 293–324.
- [8] M. W. Davis, *Some aspherical manifolds*, Duke Math. J. **55**:1 (1987), 105–139.
- [9] M. W. Davis, *Nonpositive curvature and reflection groups*, in *Handbook of Geometric Topology* (ed. by R. J. Daverman and R. B. Sher), Elsevier Science B.V., 2002, 373–422.
- [10] M. W. Davis, T. Januszkiewicz, *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Math. J. **62**:2 (1991), 417–451.
- [11] M. W. Davis, T. Januszkiewicz, *Hyperbolization of polyhedra*, J. Diff. Geom. **34** (1991), 347–388.
- [12] A. A. Gaifullin, *Realisation of cycles by aspherical manifolds*, Uspekhi Mat. Nauk **63**:3 (2008), 157–158; English transl., Russ. Math. Surv. **63**:3 (2008), 562–564; arXiv:0806.3580.
- [13] A. A. Gaifullin, *The manifold of isospectral symmetric tridiagonal matrices and realization of cycles by aspherical manifolds*, Trudy MIAN **263** (2008), 44–63, English transl., Proc. Steklov Inst. Math. **263** (2008), 38–56.
- [14] A. A. Gaifullin, *The construction of combinatorial manifolds with prescribed sets of links of vertices*, Izvestiya RAN, Ser. matem. **72**:5 (2008), 3–62; English transl., Izvestiya: Mathematics **72**:5 (2008), 845–899, arXiv:0801.4741.
- [15] M. Gromov, *Volume and bounded cohomology*, Publ. Math. I.H.E.S. **56** (1982), 5–99.
- [16] M. Gromov, *Hyperbolic groups*, Essays in Group Theory **8**, edited by S. M. Gersten, Math. Sci. Res. Inst. Publ., Springer-Verlag, New York, 1987, 75–264.
- [17] D. Kotschick, C. Löb, *Fundamental classes not representable by products*, J. London Math. Soc. **79**:3 (2009), 545–561; arXiv: 0806.4540.
- [18] F. Löbell, *Beispiele geschlossener dreidimensionaler Clifford-Kleinscher Räume negativer Krümmung*, Ber. Sachs. Acad. Wiss. **83** (1931), 168–174.
- [19] J. W. Milnor, W. P. Thurston, *Characteristic numbers of 3-manifolds*, Enseign. Math. **23** (1977), 249–254.
- [20] S. P. Novikov, *Homotopy properties of Thom complexes*, Matem. Sb. **57(99)**:4 (1962), 407–442.
- [21] P. Ontaneda, *Pinched Smooth Hyperbolization*, arXiv:1110.6374.
- [22] D. Stone, *Geodesics in piecewise linear manifolds*, Trans. Amer. Math. Soc. **215** (1976), 1–44.
- [23] R. Thom, *Quelques propriétés globales des variétés différentiables*, Comm. Math. Helv. **28** (1954), 17–86.
- [24] C. Tomei, *The topology of the isospectral manifold of tridiagonal matrices*, Duke Math. J. **51**:4 (1984), 981–996.
- [25] E. B. Vinberg, *Discrete linear groups generated by reflections*, Izvestiya AN SSSR, Ser. Matem. **35**:5 (1971), 1072–1112; English transl., Math. USSR Izvestiya **5**:5 (1971), 1083–1119.
- [26] E. B. Vinberg, *Absence of crystallographic reflection groups in Lobachevsky space of large dimension*, Trudy Moskov. Mat. Obshch. **47** (1984), 68–102; English transl., Trans. Mosc. Math. Soc. **47** (1985), 75–112.

- [27] S. Wang, *Non-zero degree maps between 3-manifolds*, Proc. of the ICM Beijing 2002, Vol. **II**, 457–468, Higher Education Press Beijing, 2002.

STEKLOV MATHEMATICAL INSTITUTE, MOSCOW

MOSCOW STATE UNIVERSITY

INSTITUTE FOR INFORMATION TRANSMISSION PROBLEMS, MOSCOW

E-mail address: `agaif@mi.ras.ru`